# MG26018 Simulation Modeling and Analysis
# 仿真建模与分析

## Lecture 3: Random Variate Generation

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# Contents

# Introduction

- Random **variable** is a variable whose values are random and depend on a probability distribution.
  - E.g., normal, exponential, Poisson, etc.

- Random **variate** is a *particular* outcome (i.e. observed sample, realization) of a random variable.
  - E.g., 5 random variates (outcomes) from a $\mathcal{N}(0, 1)$ random variable: $0.5377, 1.8339, -2.2588, 0.8622, 0.3188$.

- When simulating a system, we often need to generate random variates (e.g., interarrival time, service time) from all kinds of distributions (e.g., exponential distribution, arbitrary empirical distribution).

# Introduction

- In practice:
  - Most simulation softwares have build-in functions to generate random variates from common distributions.
  - Most programming languages have implemented the common routines of random variate generation in the libraries.

- It is nevertheless worthwhile to understand how random variate generation occurs.
  - In case when build-in functions or libraries are unavailable.
  - To better understand the randomness in stochastic simulation.
  - Be alert to some inadequate random variate generator.

- To produce a sequence of random variates from a given distribution:
  1. Start with random variates from $\text{Uniform}[0,1]$ (called **random numbers**).
  2. All random variates with given distribution are "transformed" from **random numbers**.

- **Random numbers** are a sequence of **independent** random observations from **uniform** distribution on $[0, 1]$.
  - If $U \sim \text{Uniform}[0, 1]$, then $\mathbb{E}[U] = \frac{1}{2}$, $\text{Var}(U) = \frac{1}{12}$, and its pdf is $f(u) = \begin{cases} 1, & 0 \le u \le 1, \\ 0, & \text{otherwise.} \end{cases}$
  - 10 random numbers: $0.2760, 0.6797, 0.6551, 0.1626, 0.1190, 0.4984, 0.9597, 0.3404, 0.5853, 0.2238.$
- Statistical Properties
  - Uniformity: Each value on $[0, 1]$ has equal likelihood.
  - Independence: No correlation between successive numbers.

- Uniformity

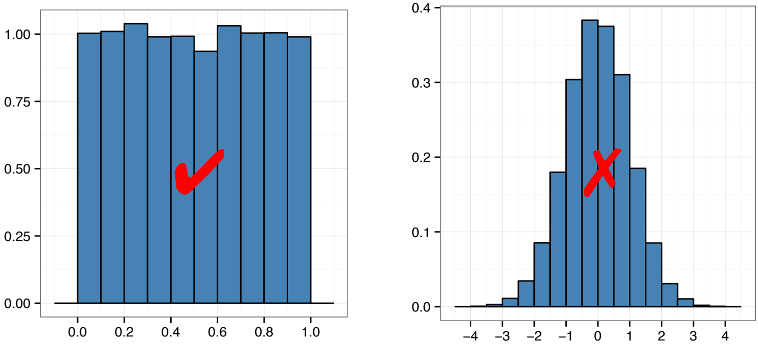



Figure: Uniformity vs Nonuniformity (*from ZHANG Xiaowei*)

- Independence

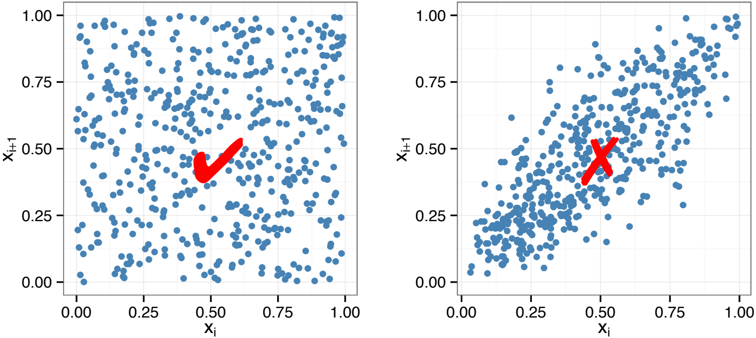



Figure: Independence vs Dependence (*from ZHANG Xiaowei*)

- A computer can NOT generate true randomness! It can only give us **pseudo-random** (伪随机) numbers.
- "Pseudo" means *false*
  - Generating random numbers by a known method removes true randomness.
  - The set of pseudo-random numbers can be repeated.
- Goal: To produce a sequence of numbers in $[0, 1]$ that imitates the ideal properties of random numbers.
  - Statistical properties are the most important.
  - True randomness is not the first priority.

- Properties of a good random number generator (RNG):
    1. Pass statistical tests.
    2. Solid theoretical support.
    3. Fast.
    4. Sufficiently long cycle (period).
    5. Portable to different computers.
    6. Replicable.

- Techniques for RNG:
    - Linear Congruential Generator (LCG)
    - Combined LCG
    - Multiple Recursive Generator (MRG)

- Linear Congruential Generator (LCG, 线性同余发生器) is a simple and early development of RNG.

1. Produce a sequence of integers $x_1, x_2, \ldots$ between $0$ and $m-1$ by

$$x_{i+1} = (ax_i + c) \bmod m, \quad i = 0, 1, 2, \ldots.$$

   - The initial value $x_0$ is called the *seed* (种子), $a$ is *multiplier* (乘子), $c$ is *increment* (增量), and $m$ is *modulus* (模数).

2. Transform $x_i$'s to values between $0$ and $1$ by

$$u_i = \frac{x_i}{m}, \quad i = 0, 1, 2, \ldots.$$

   - Possible values of $u_i$: $\{0, \frac{1}{m}, \ldots, \frac{m-1}{m}\}$. (May not cover all!)

- The selection of the values for $a$, $c$, $m$, and $x_0$ drastically affects the statistical properties and the cycle length.

- Example: Use LCG with $x_0 = 27$, $a = 17$, $c = 43$, and $m = 100$.

$$\begin{aligned}
x_0 &= 27 \\
x_1 &= (17 \times 27 + 43) \bmod 100 &= 502 \bmod 100 = 2 \\
u_1 &= 2/100 = 0.02 \\
x_2 &= (17 \times 2 + 43) \bmod 100 &= 77 \bmod 100 = 77 \\
u_2 &= 77/100 = 0.77 \\
x_3 &= (17 \times 77 + 43) \bmod 100 &= 1352 \bmod 100 = 52 \\
u_3 &= 52/100 = 0.52 \\
x_4 &= (17 \times 52 + 43) \bmod 100 &= 927 \bmod 100 = 27 \\
u_4 &= 27/100 = 0.27
\end{aligned}$$

The cycle length is only $4$!

- Try `https://xiaoweiz.shinyapps.io/randNumGen` for different parameters.

- An actual use of LCG (Lewis et al. 1969): $a = 7^5$, $c = 0$, $m = 2^{31} - 1 = 2,147,483,647$ (a prime number).
  - It adopts $u_i = \frac{x_i}{m+1}$.
  - It passes many of the standard statistical tests.
  - Cycle length $\approx 2^{31} - 2 \approx 2 \times 10^9$ (well over 2 billion).

- Note: By letting modulus $m$ be a power of 2 (or close), the modulo operation can be conducted efficiently, since most digital computers use a binary representation of numbers.

- As computing power has increased, LCG is not adequate nowadays; more sophisticated RNGs are used in practice.

- Combined LCG: Combine $J$ ($\geq 2$) LCG (with $c = 0$).
- For 32-bit computers, L'Ecuyer (1988) suggests combining $J = 2$ generators with $a_1 = 40,014$, $m_1 = 2,147,483,563$, $a_2 = 40,692$, and $m_2 = 2,147,483,399$.

1. Select seed $x_{1,0}$ in the range $[1, m_1 - 1]$ for the first generator, and seed $x_{2,0}$ in the range $[1, m_2 - 1]$ for the second. Set $j = 0$.
2. Calculate
$$x_{1,j+1} = a_1 x_{1,j} \mod m_1,$$
$$x_{2,j+1} = a_2 x_{2,j} \mod m_2.$$
3. Let $x_{j+1} = (x_{1,j+1} - x_{2,j+1}) \mod (m_1 - 1)$.
(*Remark*: $\mathrm{mod}$ uses floored division, i.e., $y \mod m = y - m\lfloor \frac{y}{m} \rfloor$.)
4. Return
$$u_{j+1} = \begin{cases} \frac{x_{j+1}}{m_1}, & \text{if } x_{j+1} > 0, \\ \frac{m_1 - 1}{m_1}, & \text{if } x_{j+1} = 0. \end{cases}$$
5. Set $j = j + 1$ and go to Step 2.

It has cycle length $(m_1 - 1)(m_2 - 1)/2 \approx 2 \times 10^{18}$.

- Multiple Recursive Generator (MRG): Extends LCG by using a higher-order recursion:

$$x_i = (a_1 x_{i-1} + a_2 x_{i-2} + \cdots + a_k x_{i-K}) \bmod m.$$

- A specific instance that has been widely implemented is `MRG32k3a`[†] (L'Ecuyer 1999), which is a *combined MRG* with $J = 2$ and $K = 3$.
  - It has cycle length $\approx 3 \times 10^{57}$, which is enormous.
  - If you could generate 2 billion $(10^9)$ pseudo-random numbers per second, then it would take longer than the age of the universe to exhaust the period of `MRG32k3a`!

[†] `MRG32k3a` or its adaptation is one of the RNGs used in MATLAB, R, SAS, Arena, etc.

- Tests based on generated sequences of numbers.
  - *Frequency Test* for uniformity (discussed in next lecture)
    – Kolmogorov–Smirnov test (柯尔莫哥洛夫– 斯米尔诺夫检验)
    – chi-square test ($\chi^2$ test, 卡方检验)
  - *Autocorrelation Test* for independence.

- There are also some *theoretical tests* without actually generating any numbers, e.g., spectral test (谱检验).

- Fortunately, the well-known RNGs which are widely used in simulation softwares and languages have been extensively tested and validated.

- Be careful when the RNG at hand is not explicitly known or documented!
  - Even RNGs that have been used for years in popular commercial softwares (e.g., Excel, Visual Basic), have been found to be inadequate (L'Ecuyer 2001).

# Random Variate Generation

- Assumption: RNG is available, i.e. we have a sequence of random numbers $(\mathrm{Uniform}[0, 1])$.
- Goal: Produce random variates from a given probability distribution (e.g. exponential, Poisson, etc.).
- Widely-used techniques†
  - Inverse-transform technique (generic)
  - Acceptance-rejection technique (generic)
  - Other ad-hoc methods for some specific distributions

† Methods introduced in this lecture are exact; there are also approximation methods such as MCMC.

- Let $F(x)$ be the cumulative distribution function (cdf) of $X$, i.e., $F(x) = \mathbb{P}(X \leq x)$.

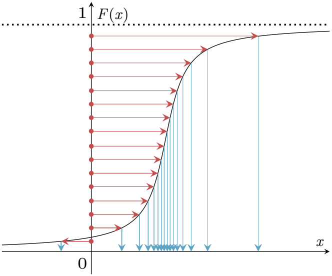



Figure: Continuous Random Variable

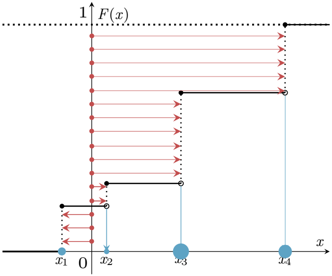



Figure: Discrete Random Variable

- Procedures
  1. Generate (as needed) random numbers (on vertical axis).
  2. Map inversely to points on horizontal axis, which are the desired random variates from $F(x)$.

- The formal definition of inverse function is
$$F^{-1}(y) := \min\{x : F(x) \geq y\}, \quad 0 \leq y \leq 1.$$

- If $U \sim \mathrm{Uniform}[0,1]$, then $F^{-1}(U)$ has the same distribution as $X$, i.e.,
$$\mathbb{P}(F^{-1}(U) \leq x) = \mathbb{P}(U \leq F(x)) = F(x).$$

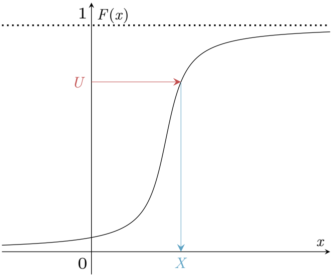



Figure: Continuous Random Variable

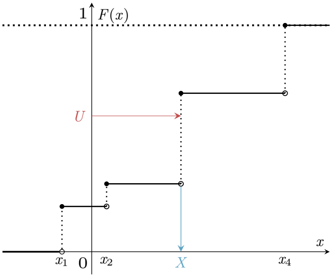



Figure: Discrete Random Variable

- The inverse-transform technique is useful when the cdf is so simple that its inverse function can be *analytically solved* or *easily computed*.
- It can be used to sample from various continuous distributions
  - uniform
  - exponential
  - triangular
  - Weibull
  - Cauchy
  - Pareto
- It can be used to sample from all (in principle) discrete distributions, e.g.,
  - discrete uniform
  - geometric
  - arbitrary empirical distribution

- Goal: Generate random variates from $X \sim \text{Uniform}[a, b]$.
- Intuition: Since $X = a + (b - a)U$, we just need to:
  1. Generate random number $u_i$;
  2. Output $x_i = a + (b - a)u_i$ as the required random variates.
- For $X \sim \text{Uniform}[a, b]$, the pdf and cdf are

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b, \\ 0, & \text{otherwise,} \end{cases} \quad F(x) = \begin{cases} 0, & x < a, \\ \frac{x-a}{b-a}, & a \le x \le b, \\ 1, & b < x. \end{cases}$$

- Solve the inverse function of $F(x)$,

$$F^{-1}(y) = a + (b - a)y, \quad 0 \le y \le 1.$$

- So, $F^{-1}(U) = a + (b - a)U$ has the same distribution as $X$.

- Goal: Generate random variates from $X \sim \text{Exp}(\lambda)$.
- For $X \sim \text{Exp}(\lambda)$, the pdf and cdf are

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0, \end{cases} \quad F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

- Solve the inverse function of $F(x)$,

$$F^{-1}(y) = -\frac{1}{\lambda} \ln(1-y), \quad 0 \leq y \leq 1.$$

- So, $F^{-1}(U) = -\frac{1}{\lambda} \ln(1-U)$ has the same distribution as $X$.
- *Remark*: $1 - U \sim \text{Uniform}[0,1] \Rightarrow -\frac{1}{\lambda} \ln(U)$ is sufficient.
- Numerical test for $\text{Exp}(1)$ in Excel.
  1. Generate 200 random numbers.
  2. Obtain 200 random variates via the inverse function.

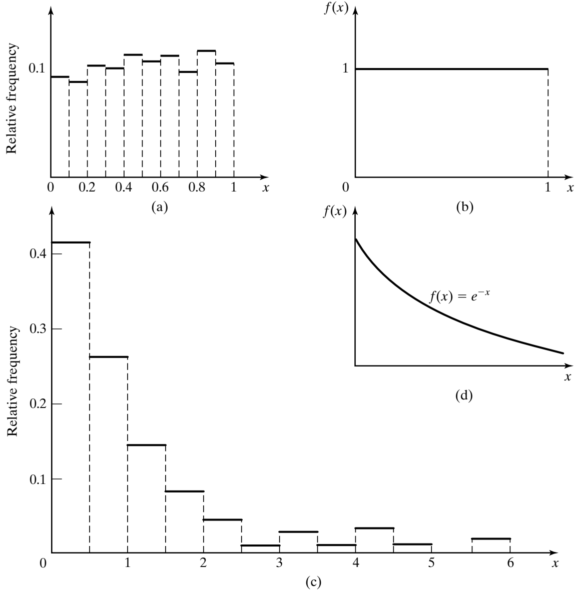



Figure:
(a) Empirical histogram of 200 generated uniform random numbers;
(b) Theoretical density of $\text{Uniform}[0,1]$;

(c) Empirical histogram of 200 generated exponential variates ($\lambda = 1$);
(d) Theoretical density of $\text{Exp}(1)$.

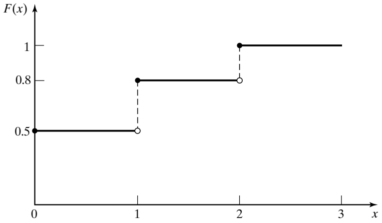
(*from* *Banks et al. (2010)*)

- Consider a discrete random variable $X$ taking values $0$, $1$, $2$ with probability $0.5$, $0.3$ and $0.2$.

- The pmf and cdf are

$$p(x) = \begin{cases} 0.5, & x = 0, \\ 0.3, & x = 1, \\ 0.2, & x = 2, \end{cases} \qquad F(x) = \begin{cases} 0, & x < 0, \\ 0.5, & 0 \le x < 1, \\ 0.8, & 1 \le x < 2, \\ 1, & 2 \le x. \end{cases}$$

- Solve the inverse function. (Recall $F^{-1}(y) := \min\{x : F(x) \ge y\}$.)




$$F^{-1}(y) = \begin{cases} 0, & 0 \le y \le 0.5, \\ 1, & 0.5 < y \le 0.8, \\ 2, & 0.8 < y \le 1. \end{cases}$$

Try it in Excel.

- Why do we need another method when the inverse-transform technique is already generic?
  - The cdf of a desired distribution may not have an analytical form.
  - The inverse cdf may not exist in closed form and may be challenging to evaluate, e.g., beta, gamma, normal, etc.
  - Although you can solve the inverse transform via numerical methods anyway, the efficiency may be low.
  - E.g., consider a pdf $f(x) = 6x(1-x)$ for $0 \leq x \leq 1$, then the cdf is $F(x) = 3x^2 - 2x^3$. Computing inverse cdf requires to solve $3x^2 - 2x^3 = y$ for given $y$.

- Acceptance-rejection technique is also useful for generating a *non-stationary Poisson process* (more details later).

- Goal: Generate random variates from $X \sim \text{Uniform}[1/4, 1]$ using acceptance-rejection technique.
  1. Generate a random number $u$ (from $U \sim \text{Uniform}[0, 1]$).
  2. If $u \geq 1/4$, **accept** $u$, output $u$ as the desired random variate; if $u < 1/4$, **reject** $u$, and return to Step 1.
  3. If another $\text{Uniform}[1/4, 1]$ random variate is needed, repeat the procedure from Step 1; stop otherwise.

- Important Observation 1: To produce one random variate using A-R technique, one may need to generate multiple random numbers.
  - Whereas there exists a one-to-one mapping for the inverse-transform method.

- Important Observation 2: The accepted values of $U$ are conditioned values.
  - $U$ itself does not have the desired distribution.
  - $U$ conditioned on the event $\{U \geq 1/4\}$ does!

- For $1/4 \leq x \leq 1$,

$$\mathbb{P}\{U \leq x | U \geq 1/4\} = \frac{\mathbb{P}\{U \leq x \text{ and } U \geq 1/4\}}{\mathbb{P}\{U \geq 1/4\}} = \frac{x - 1/4}{3/4},$$

which is exactly the desired cdf of $X \sim \mathrm{Uniform}[1/4, 1]$.

- Suppose we want to generate random variates from $X$, whose density $f(x)$ has support $[a, b]$ and is upper bounded by $M$.

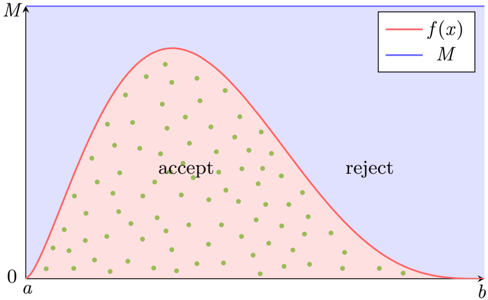



Figure: Bounded Support (*original image from ZHANG Xiaowei*)

1. Generate random variate pairs $(y_1, z_1)$, $(y_2, z_2)$, ..., from $\text{Uniform}\{(y, z) : a \le y \le b,\ 0 \le z \le M\}$.
   - $y_i$ from $Y \sim \text{Uniform}[a, b]$, $z_i$ from $Z \sim \text{Uniform}[0, M]$
2. Accept the pair if $z_i < f(y_i)$ and output $y_i$ as random variate from $X$ with density $f(x)$.

- $Y$ conditioned on the event $\{Z < f(Y)\}$ has the same distribution as $X$, i.e., having density $f(x)$.
  - $(Y, Z) \sim \text{Uniform}\{(y, z) : a \le y \le b,\ 0 \le z \le M\}$.

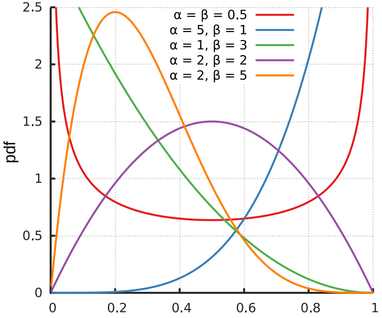
*Proof.*

$$
\begin{aligned}
\mathbb{P}\{Y \le x | Z < f(Y)\} &= \frac{\mathbb{P}\{Y \le x, Z < f(Y)\}}{\mathbb{P}\{Z < f(Y)\}} \\
&= \frac{\int_a^x \int_0^{f(y)} f_{Y,Z}(y,z)\mathrm{d}z\mathrm{d}y}{\int_a^b \int_0^{f(y)} f_{Y,Z}(y,z)\mathrm{d}z\mathrm{d}y} \\
&= \frac{\int_a^x \int_0^{f(y)} \frac{1}{(b-a)M}\mathrm{d}z\mathrm{d}y}{\int_a^b \int_0^{f(y)} \frac{1}{(b-a)M}\mathrm{d}z\mathrm{d}y} = \frac{\int_a^x \int_0^{f(y)} \mathrm{d}z\mathrm{d}y}{\int_a^b \int_0^{f(y)} \mathrm{d}z\mathrm{d}y} \\
&= \frac{\int_a^x f(y)\mathrm{d}y}{\int_a^b f(y)\mathrm{d}y} = \frac{\mathbb{P}\{X \le x\}}{1} = \mathbb{P}\{X \le x\}. \quad \blacksquare
\end{aligned}
$$

Note: $f_{Y,Z}(y,z) = \dfrac{1}{(b-a)M}$

- The acceptance rate is $\mathbb{P}\{Z < f(Y)\} = \frac{1}{(b-a)M}$.

- Goal: Generate random variates from $\mathrm{Beta}(\alpha, \beta)$, where the density is $f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$, $x \in [0, 1]$.




- If $\alpha > 1$ and $\beta > 1$, then $f(x)$ is maximized at $x = \frac{\alpha-1}{\alpha+\beta-2}$ and the maximum is $M = \frac{(\alpha-1)^{\alpha-1}(\beta-1)^{\beta-1}}{(\alpha+\beta-2)^{\alpha+\beta-2}B(\alpha,\beta)}$.
- The acceptance rate is $\frac{1}{(b-a)M} = \frac{1}{(1-0)M} = \frac{1}{M}$.

- Generate random variates from $X$, whose density $f(x)$ is upper bounded by $Mg(x)$, where $g(x)$ is *instrumental* density.

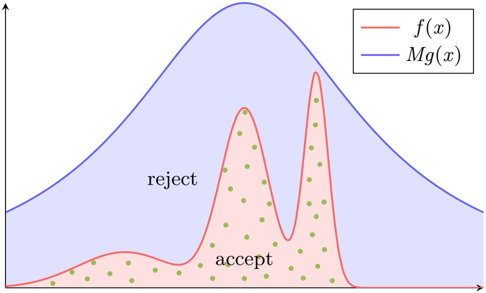



Figure: Unbounded Support (*original image from ZHANG Xiaowei*)

1. Generate random variate pairs $(y_1, z_1)$, $(y_2, z_2)$, ..., from $\text{Uniform}\{(y,z) : y \in \text{support of } g(\cdot),\ 0 \le z \le Mg(y)\}$.
   - $y_i$ from $Y \sim g(\cdot)$, $z_i$ from $Z \sim \text{Uniform}[0, Mg(y_i)]$ **(why?)**
2. Accept the pair if $z_i < f(y_i)$ and output $y_i$ as random variate from $X$ with density $f(x)$.

- $Y$ conditioned on the event $\{Z < f(Y)\}$ has the same distribution as $X$, i.e., having density $f(x)$.
  - Let $\Theta$ denote $\{(y, z) : y \in \text{support of } g(\cdot),\ 0 \le z \le Mg(y)\}$.
  - $(Y, Z) \sim \mathrm{Uniform}\,\Theta$.

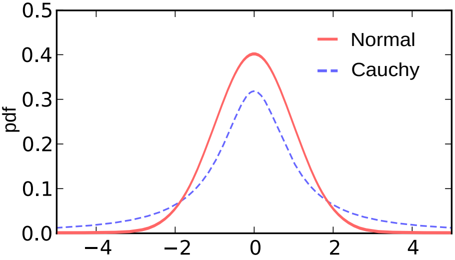
*Proof.*

$$\begin{aligned}\mathbb{P}\{Y \le x | Z < f(Y)\} &= \frac{\mathbb{P}\{Y \le x, Z < f(Y)\}}{\mathbb{P}\{Z < f(Y)\}} \\ &= \frac{\int_{-\infty}^{x}\int_0^{f(y)} f_{Y,Z}(y,z)\mathrm{d}z\mathrm{d}y}{\int_{-\infty}^{\infty}\int_0^{f(y)} f_{Y,Z}(y,z)\mathrm{d}z\mathrm{d}y} \\ &= \frac{\int_{-\infty}^{x}\int_0^{f(y)} \frac{1}{\Theta\text{ area}}\mathrm{d}z\mathrm{d}y}{\int_{-\infty}^{\infty}\int_0^{f(y)} \frac{1}{\Theta\text{ area}}\mathrm{d}z\mathrm{d}y} = \frac{\int_{-\infty}^{x}\int_0^{f(y)} \mathrm{d}z\mathrm{d}y}{\int_{-\infty}^{\infty}\int_0^{f(y)} \mathrm{d}z\mathrm{d}y} \\ &= \frac{\int_{-\infty}^{x} f(y)\mathrm{d}y}{\int_{-\infty}^{\infty} f(y)\mathrm{d}y} = \frac{\mathbb{P}\{X \le x\}}{1} = \mathbb{P}\{X \le x\}. \quad \blacksquare\end{aligned}$$

Note: $f_{Y,Z}(y,z) = \frac{1}{\Theta\text{ area}}$

- The acceptance rate is
  $\mathbb{P}\{Z < f(Y)\} = \frac{1}{\Theta\text{ area}} = \frac{1}{\int_{-\infty}^{\infty} Mg(y)\mathrm{d}y} = \frac{1}{M\int_{-\infty}^{\infty} g(y)\mathrm{d}y} = \frac{1}{M}$.

- Goal: Generate random variates from $\mathcal{N}(0,1)$, where the density is $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, $x \in (-\infty, \infty)$.
- Use $\mathrm{Cauchy}(0)$ density as instrumental density, which is $g(x) = \frac{1}{\pi(1+x^2)}$, $x \in (-\infty, \infty)$.




- It is easy to see that $\frac{f(x)}{g(x)} = \sqrt{\frac{\pi}{2}}(1+x^2)e^{-\frac{x^2}{2}}$ is maximized at $x = \pm 1$ and the maximum is $\sqrt{\frac{2\pi}{e}}$, which is the required $M$.
- The acceptance rate is $\frac{1}{M} = \sqrt{\frac{e}{2\pi}} \approx 0.6577$.

- **Univariate normal**: A normal RV with mean $\mu$ and s.d. $\sigma$ has pdf

$$\phi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in (-\infty, \infty).$$

- If $\mu = 0$ and $\sigma = 1$, then it is a *standard* normal RV.
- If $Z \sim \mathcal{N}(0,1)$, then $\mu + \sigma Z \sim \mathcal{N}(\mu, \sigma^2)$.
- Generate $\mathcal{N}(0,1)$ random variate
  - Method 1 Acceptance-rejection technique (from Cauchy).
  - Method 2 Box–Muller method.

- Box–Muller method
  1. Generate $u_1$ and $u_2$ independently from $\mathrm{Uniform}[0, 1]$.
  2. Let $z_1 = \sqrt{-2 \ln u_1} \cos(2\pi u_2)$ and $z_2 = \sqrt{-2 \ln u_1} \sin(2\pi u_2)$.

- $z_1$ and $z_2$ are random variates from $\mathcal{N}(0, 1)$ (independent).

- Intuition:
  - For two independent $\mathcal{N}(0, 1)$ RVs $Z_1$ and $Z_2$,
  $$Z_1^2 + Z_2^2 \sim \chi_2^2.$$
  - $X \sim \mathrm{Exp}(1/2) \;\Leftrightarrow\; X \sim \chi_2^2$.
  - $-2 \ln u_1$ is a random variate from $\mathrm{Exp}(1/2)$ (and thus $\chi_2^2$).
  - The angle is distributed uniformly around the circle.

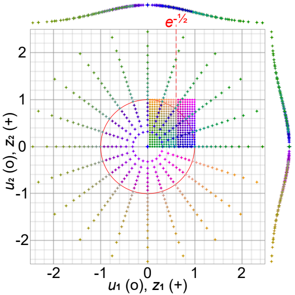



Figure: Box–Muller Method Visualisation
(image by Cmglee / CC BY 3.0)
Interactive Graph: Wikimedia Backup

- **Multivariate normal**: Univariate normal $Z_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$, $i = 1, \ldots, d$, with $\Sigma_{ij} := \mathrm{Cov}(Z_i, Z_j)$, form a random vector $\boldsymbol{Z} = (Z_1, \ldots, Z_d)^\intercal \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, and it has joint pdf

$$\phi(\boldsymbol{x}) = \frac{1}{(2\pi)^{d/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\left\{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^\intercal \boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right\},$$

  $\boldsymbol{x} \in \mathbb{R}^d$, where $|\boldsymbol{\Sigma}|$ is the determinant of $\boldsymbol{\Sigma}$.
  - $\boldsymbol{\Sigma} = (\Sigma_{ij})$ is a symmetric and positive semidefinite matrix.
- If $\mu_i = 0$ and $\sigma_i = 1$ for all $i$, and $\Sigma_{ij} = 0$ for $i \neq j$ (pairwise independence), then $\boldsymbol{Z} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I})$.
- If $\boldsymbol{Z} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I})$, and $\boldsymbol{\Sigma} = \boldsymbol{A}\boldsymbol{A}^\intercal$ (Cholesky decomposition), then $\boldsymbol{\mu} + \boldsymbol{A}\boldsymbol{Z} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.
- There are many other relationships among various probability distributions.
  - See, for example, Leemis & McQueston (2008) and the interactive graph `http://www.math.wm.edu/~leemis/chart/UDR/UDR.html`

- **Poisson process** with rate $\lambda$: Interarrival time distribution is exponential with rate $\lambda$ (or mean $1/\lambda$), and

$$N(t+h) - N(t) \sim \text{Poisson}(\lambda h). \quad \text{(same as } N(h)\text{)}$$

- To generate Poisson process with rate $\lambda$, one only need to generate iid $\text{Exp}(\lambda)$ random variates.
  - $s_i$, the arrival time of the $i$th arrival, satisfies

$$s_i = s_{i-1} - (1/\lambda)\ln(u_i), \; i = 1, 2, \ldots.$$

- **Nonhomogeneous Poisson process** with rate (intensity) function $\lambda(t)$:

$$N(t+h) - N(t) \sim \text{Poisson}(m(t+h) - m(t)),$$

where $m(t) = \int_0^t \lambda(s)\mathrm{d}s$.

- To generate nonhomogeneous Poisson process with rate function $\lambda(t)$, one can use the acceptance-rejection method (which is also called *thinning* in this context).
- Idea behind thinning:
  - Generate a *stationary* Poisson arrival process at the fastest rate $\lambda^* = \max_t \lambda(t)$.
  - But "accept" only a portion of arrivals, thinning out just enough to get the desired time-varying rate.
- Algorithm:
  1. Set $t = 0$ and $i = 1$.
  2. Generate $x$ from $\mathrm{Exp}(\lambda^*)$, and let $t \leftarrow t + x$ (this is the arrival time of the *stationary* Poisson process with rate $\lambda^*$).
  3. Generate random number $u$ (from $\mathrm{Uniform}[0, 1]$). If $u \leq \lambda(t)/\lambda^*$, then $s_i = t$ and $i \leftarrow i + 1$.
  4. Go to Step 2.